IX. Instability

Exercise I. Let $\boldsymbol{U} = U(y)\boldsymbol{x}$ be the parallel shear flow of an incompressible viscous fluid. Suppose that it is superposed by a small three-dimensional disturbance of a form:

$$\bar{u}_1 = \mathrm{Real}\left[\hat{\bar{u}}(y)e^{i(\alpha x + \beta z - \omega t)}\right].$$

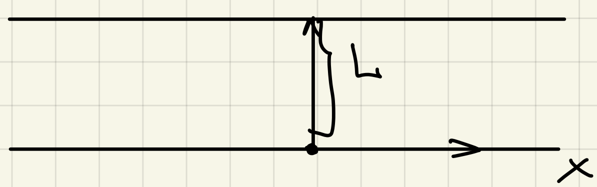
Derive the perturbation equations in a form:

$$\left[(-i\omega + i\alpha U)\left(\frac{d^2}{dy^2} - k^2\right) - i\alpha U'' - \frac{1}{Re}\left(\frac{d^2}{dy^2} - k^2\right)^2\right]\hat{v} = 0, \quad (1)$$

$$\left[(-i\omega + i\alpha U) - \frac{1}{Re}\left(\frac{d^2}{dy^2} - k^2\right)\right]\hat{\xi} = -i\beta U'\hat{v}, \quad (2)$$

where $\xi = \underbrace{(\bar{\nabla} \times \bar{u})}_{\text{vorticity}}{}_y$ and $k^2 = \alpha^2 + \beta^2$.

Equations (1) and (2) are known as the Orr-Sommerfeld and Squire equations, respectively.



$$\frac{d\bar{u}_1}{dt} = L\bar{u}_1 \quad \text{– linearized}$$

A formal solution: $\bar{u}_1(t) = \bar{u}_1(0)\, e^{Lt}$

$$e^{\lambda_{max} t} \le \| e^{Lt} \| = \| Q e^{\Lambda t} Q^{-1} \|$$

$$\le \|Q\| \|Q^{-1}\| e^{\lambda_{max} t}$$

$t \gg 1$, $\bar{u}_1 \sim e^{\lambda_{max} t}$

$$u_1 = \mathrm{Real}\left[\hat{u}(y)\, e^{i(\alpha x + \beta z - \omega_r t - i\omega_i t)}\right]$$

$$= |\hat{u}(y)|\, e^{\omega_i t}\, \underbrace{\mathrm{Real}\left[e^{i(\alpha x + \beta z - \omega_r t)}\right]}_{\cos(\alpha x + \beta z - \omega_r t)}$$

If $\omega_i > 0 \rightarrow |u_1|$ grows exponentially

instability

Part I. What are the equations governing fluid motion in the absence of disturbance?

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\bar{\nabla})\bar{u} = = -\frac{1}{\rho}\bar{\nabla} p + \nu \bar{\nabla}^2 \bar{u},$$

$$\bar{\nabla}\cdot\bar{u} = 0.$$

Subs. $\bar{U} = U(y)\bar{e}_x, \quad P = P(x), \quad \frac{\partial}{\partial x}P = \text{const}$

$$\nu U'' - \frac{1}{\rho}\frac{\partial P}{\partial x} = 0$$

Part II. Write down the equations of motion for the perturbed flow without assuming any particular form of a perturbation. Linearize the equations of motion with respect to perturbation variables.

$$\bar{u} = \bar{U}(y) + \bar{u}_1(x,y,z,t),$$
$$p = P + p_1(x,y,z,t).$$

$$\frac{\partial \bar{u}_1}{\partial t} + U\frac{\partial \bar{u}_1}{\partial x} + v_1 U'\bar{e}_x = -\frac{1}{\rho}\bar{\nabla} p_1 + \nu\bar{\nabla}^2\bar{u}_1,$$

$$\bar{\nabla}\cdot\bar{u}_1 = 0.$$

$$\bar{u}\cdot\bar{\nabla} u = \left(U\frac{\partial}{\partial x} + u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y} + w_1\frac{\partial}{\partial z}\right)(U+u_1)$$
$$= v_1 U' + U\frac{\partial u_1}{\partial x}$$

$$O(v_1 U') \sim \frac{|v_1||U|}{L} \sim \frac{\varepsilon U^2}{L}$$

$$O\left(u_1\frac{\partial u_1}{\partial x}\right) \sim \frac{\varepsilon^2 U^2}{L}$$

small parameter

Part III. How can pressure be eliminated from the equations?

$$\nabla\cdot\left(\frac{\partial \bar{u}_1}{\partial t} + U\frac{\partial \bar{u}_1}{\partial x} + v_1 U' \bar{e}_x\right) = \nabla\cdot\left(-\frac{1}{\rho}\nabla p_1 + \nu\nabla^2\bar{u}_1\right),$$

$$\nabla\cdot\left(U\frac{\partial \bar{u}}{\partial x} + v_1 U'\bar{e}_x\right) = -\frac{1}{\rho}\nabla^2 p_1$$

$$\frac{\partial \bar{u}}{\partial x}\cdot\nabla U + U'\frac{\partial v_1}{\partial x} = \boxed{2U'\frac{\partial v_1}{\partial x} = -\frac{1}{\rho}\nabla^2 p_1}$$

Poisson eq-n for pressure

Part IV. Derive the Orr-Sommerfeld equation.

$$\frac{\partial v_1}{\partial t} + U\frac{\partial v_1}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v_1$$

$$\frac{\partial}{\partial t}\nabla^2 v_1 + \underbrace{\nabla^2\left(U\frac{\partial v_1}{\partial x}\right)}_{U''\frac{\partial v_1}{\partial x} + 2U'\frac{\partial^2 v_1}{\partial x\partial y} + U\frac{\partial}{\partial x}\nabla^2 v_1} = -\frac{1}{\rho}\nabla^2\frac{\partial p}{\partial y} + \nu\nabla^4 v_1$$

$$-\frac{1}{\rho}\nabla^2\frac{\partial p}{\partial y} = 2\frac{\partial}{\partial y}\left(U'\frac{\partial v_1}{\partial x}\right) = 2U''\frac{\partial v_1}{\partial x} + 2U'\frac{\partial^2 v_1}{\partial x\partial y}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 v_1 + U''\frac{\partial v_1}{\partial x} + \cancel{2U'\frac{\partial^2 v_1}{\partial x\partial y}} = 2U''\frac{\partial v_1}{\partial x} + \cancel{2U'\frac{\partial^2 v_1}{\partial x\partial y}} + \nu\nabla^4 v_1$$

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 - U''\frac{\partial}{\partial x} - \nu\nabla^4\right]v_1 = 0$$

Non-dimensionalization:

$$\begin{cases} \underbrace{v_1}_{\text{dim.}} \to \underbrace{v_1}_{\text{non-dim.}} \overset{\text{some charact. velocity}}{U_0}, \quad U \to U U_0, \\ \bar{x} \to \bar{x} L, \quad t \to t\frac{L}{U_0} \end{cases}$$

$$\frac{U_0^2}{L^3}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 v_1 - \frac{U_0^2}{L^3}U''\frac{\partial v_1}{\partial x} - \frac{\nu U_0}{L^4}\nabla^4 v_1 = 0$$

Multiply with $\frac{L^3}{U_0^2}$:

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\bar{\nabla}^2 v_1 - U''\frac{\partial v_1}{\partial x} - \frac{1}{Re}\bar{\nabla}^4 v_1\right] = 0,$$

$$Re = \frac{U_0 L}{\nu}$$

Assume $v_1 = \hat{v}(y)\,e^{i(\alpha x + \beta z - \omega t)}$

$$(-i\omega + i\alpha U)\left(\frac{d^2}{dy^2} - \alpha^2 - \beta^2\right)\hat{v}\,\cancel{e^{i(\alpha x+\beta z-\omega t)}} - i\alpha U''\hat{v}\,\cancel{e^{i(\alpha x+\beta z-\omega t)}} - \frac{1}{Re}\left(\frac{d^2}{dy^2} - \alpha^2 - \beta^2\right)^2\hat{v} \times \cancel{e^{i(\alpha x+\beta z-\omega t)}} = 0$$

$$k^2 = \alpha^2 + \beta^2$$

Orr-Sommerfeld equation:

$$\left[(-i\omega + i\alpha U)\left(\frac{d^2}{dy^2} - k^2\right) - i\alpha U'' - \frac{1}{Re}\left(\frac{d^2}{dy^2} - k^2\right)^2\right]\hat{v} = 0$$

Part V. Derive the Squire equation.

$$\bar{\nabla}\times\left(\frac{\partial\bar{u}}{\partial t} + \bar{u}\cdot\bar{\nabla}\bar{u}\right) = \bar{\nabla}\times\left(-\bar{\nabla}p + \frac{1}{Re}\bar{\nabla}^2\bar{u}\right)$$

$$\frac{\partial\bar{\xi}}{\partial t} + \bar{\nabla}\times(\bar{u}\cdot\bar{\nabla}\bar{u}) = \frac{1}{Re}\bar{\nabla}^2\bar{\xi},$$

where $\bar{\nabla}\times(\bar{u}\cdot\bar{\nabla}\bar{u}) = \bar{\nabla}\times(\bar{\xi}\times\bar{u}) = \bar{u}\cdot\bar{\nabla}\bar{\xi} - \bar{\xi}\cdot\bar{\nabla}\bar{u}$.

$$\bar{u}\cdot\bar{\nabla}\xi_y = \left(U\frac{\partial}{\partial x} + u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y}\right)\xi_{1,y} = U\frac{\partial\xi_{1,y}}{\partial x}$$

$$\bar{\xi}\cdot\bar{\nabla}v = \left(\xi_{1,x}\frac{\partial}{\partial x} + \xi_{1,y}\frac{\partial}{\partial y} - U'\frac{\partial}{\partial z} + \xi_{1,z}\frac{\partial}{\partial z}\right)v_1$$

$$= -U'\frac{\partial v_1}{\partial z}$$

$$\Rightarrow \frac{\partial\xi_{1,y}}{\partial t} + U\frac{\partial\xi_{1,y}}{\partial x} - \frac{1}{Re}\bar{\nabla}^2\xi_{1,y} = -U'\frac{\partial v_1}{\partial z}$$

As we are left with only y-component of pert. vorticity $\bar{\xi}_1$, we change notation:

$$\xi_{1,y} \rightarrow \xi$$

If $\xi = \hat{\xi}(y)\, e^{i(\alpha x + \beta z - \omega t)}$,

$$(-i\omega + i\alpha U)\hat{\xi} - \frac{1}{Re}\left(\frac{d^2}{dy^2} - k^2\right)\hat{\xi} = -i\beta U' \hat{v}$$

Divide by $i\alpha$:

$$\underbrace{(U - c)\hat{\xi} - \frac{1}{i\alpha Re}\left(\frac{d^2}{dy^2} - k^2\right)\hat{\xi} = -\frac{\beta}{\alpha} U' \hat{v}}_{\text{Squire eq-n.}}$$

The BC:

$\bar{u}_1 = 0$ at $y = \pm L/2 \Rightarrow \hat{v} = \hat{\xi} = 0$ at $y = \pm L/2$

We need 2 more BC:

$$\bar{\nabla}\cdot\bar{u}_1 = 0 \Rightarrow i\alpha \underbrace{\hat{u}}_{\substack{0 \text{ at} \\ y=\pm L/2}} + \hat{v}' + i\beta \underbrace{\hat{w}}_{\substack{0 \text{ at} \\ y=\pm L/2}} = 0$$

$$\Rightarrow \frac{d\hat{v}}{dy} = 0 \text{ at } y = \pm L/2$$

The Squire theorem

$$\left[(-i\omega + i\alpha U)\left(\frac{d^2}{dy^2} - k^2\right) - i\alpha U'' - \frac{1}{Re}\left(\frac{d^2}{dy^2} - k^2\right)^2\right]\hat{v} = 0$$

$$\left[(-i\omega + i\alpha U) \quad - \frac{1}{Re}\left(\frac{d^2}{dy^2} - k^2\right)\right]\hat{\xi} = -i\beta U'\hat{v},$$

Phase velocity $c = \omega / \alpha$.

$$\left[(U - c)\left(\frac{d^2}{dy^2} - k^2\right) - U'' - \frac{1}{i\alpha Re}\left(\frac{d^2}{dy^2} - k^2\right)^2\right]\hat{v} = 0,$$

$$\left[(U - c) - \frac{1}{i\alpha Re}\left(\frac{d^2}{dy^2} - k^2\right)\right]\hat{\xi} = -\frac{\beta}{\alpha}U'\hat{v}.$$

Largest imag. part of αc.

$$A\begin{pmatrix}\hat{v}\\ \hat{\xi}\end{pmatrix} = c\,B\begin{pmatrix}\hat{v}\\ \hat{\xi}\end{pmatrix}$$

Orr-Sommerfeld modes: $\{\hat{v}_n, \hat{\xi}_n, c_n\}$

Squire modes: $[\hat{v} = 0, \hat{\xi}_m, c_m\}$

Th. about damped Squire modes: $\forall\, \mathrm{Im}(c_m) < 0$.

$\rightarrow$ Squire modes are not interesting for stability.

We only solve the Orr-Som.

$$\left[(U - c)\left(\frac{d^2}{dy^2} - k^2\right) - U'' - \frac{1}{i\alpha Re}\left(\frac{d^2}{dy^2} - k^2\right)^2\right]\hat{v} = 0,$$

$$\left[(U - c)\left(\frac{d^2}{dy^2} - \alpha_{2D}^2\right) - U'' - \frac{1}{i\alpha_{2D} Re_{2D}}\left(\frac{d^2}{dy^2} - \alpha_{2D}^2\right)\right]\hat{v} = 0,$$

where $\alpha_{2D} = k = \sqrt{\alpha^2 + \beta^2}$,

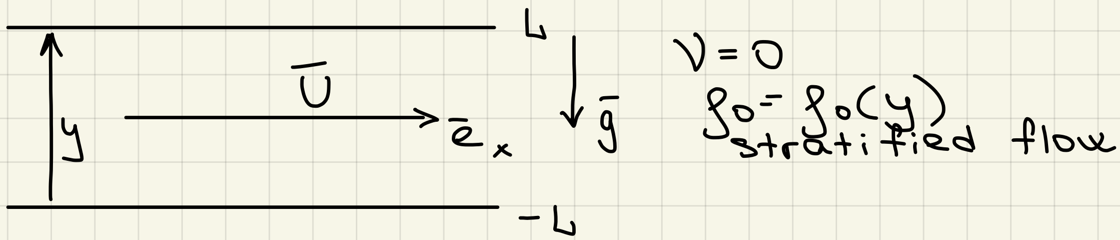
$$Re_{2D} = Re\frac{\alpha}{\alpha_{2D}} = Re\frac{\alpha}{k} < Re$$

The 2 eq.s are mathem. identical $\rightarrow$ each 3D solution $\{\hat{v}; c, Re\}$ has a corresponding 2D solution $\{\hat{V}_{2D}; c_{2D}, Re_{2D}\}$ that is less stable as

$$Re_{2D} < Re.$$

9.2 It may be shown that a small 2D perturbation to a shear flow of an inviscid stratified fluid is governed by an equation:

$$\hat{v}'' + \frac{\rho_0'}{\rho_0}\hat{v}' + \left[\frac{N^2}{(c-U)^2} + \frac{U''}{c-U} + \frac{\rho_0'}{\rho_0}\frac{U'}{c-U} - k^2\right]\hat{v} = 0.$$



$$\bar{u} = U(y)\bar{e}_x + \bar{u}_1(\bar{x}, t)$$
$$p = P + p_1(\bar{x}, t)$$
$$\rho = \rho_0(y) + \rho_1(\bar{x}, t),$$

complex

where $\bar{u}_1(\bar{x}, t) = \text{Real}\left[\hat{\bar{u}}(y)\, e^{i(kx - \omega t)}\right]$

$$\Rightarrow \sim e^{\omega_i t} \cos(kx - \omega_r t)$$

Part I. Derive the perturbation equation.

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\cdot\bar{\nabla}\bar{u}\right) = -\bar{\nabla} p + \rho\bar{g},$$

$$\bar{\nabla}\cdot\bar{u} = 0,$$

$$\frac{\partial \rho}{\partial t} + \bar{u}\cdot\bar{\nabla}\rho = 0.$$

I. In the absence of a pert.

1.x) $\frac{\partial P}{\partial x} = 0,$

1.y) $\frac{\partial P}{\partial y} + \rho_0 g = 0$

2) $U\frac{\partial \rho_0}{\partial x} = 0$

II. Perturbed flow

1.x) $(\rho_0 + \rho_1)\left[\frac{\partial u_1}{\partial t} + \left(U\frac{\partial}{\partial x} + u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y}\right)(U + u_1)\right]$

$$= -\cancel{\frac{\partial P}{\partial x}} - \frac{\partial p_1}{\partial x}$$

Linearization $\rho_0\left(\frac{\partial u_1}{\partial t} + v_1\frac{\partial U}{\partial y} + U\frac{\partial u_1}{\partial x}\right) = -\frac{\partial p_1}{\partial x}$

1. y) $(\rho_0+\rho_1)\left[\frac{\partial v_1}{\partial t}+\left(U\frac{\partial}{\partial x}+u_1\frac{\partial}{\partial x}+v_1\frac{\partial}{\partial y}\right)v_1\right]$

$$=-\cancel{\frac{\partial P}{\partial y}}-\frac{\partial p_1}{\partial y}-\cancel{\rho_0 g}-\rho_1 g$$

$$\Rightarrow \rho_1\left(\frac{\partial v_1}{\partial t}+U\frac{\partial v_1}{\partial x}\right)=-\frac{\partial p_1}{\partial y}-\rho_1 g$$

2) $\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}=0$

3) $\frac{\partial \rho_1}{\partial t}+\left(U\frac{\partial}{\partial x}+u_1\frac{\partial}{\partial x}+v_1\frac{\partial}{\partial y}\right)(\rho_0+\rho_1)=0$

$$\rightarrow \frac{\partial \rho_1}{\partial t}+v_1\frac{\partial \rho_0}{\partial y}+U\frac{\partial \rho_1}{\partial x}=0$$

Compute $\frac{\partial}{\partial y}(1.x)-\frac{\partial}{\partial x}(1.y)$

$$\rho_0'\left(\frac{\partial u_1}{\partial t}+v_1U'+U\frac{\partial u_1}{\partial x}\right)+\rho_0\left(\frac{\partial^2 u_1}{\partial t\partial y}+\cancel{\frac{\partial v_1}{\partial y}U'}\right.$$
$$\left.+v_1U''+\cancel{U'\frac{\partial u_1}{\partial x}}+U\frac{\partial^2 u_1}{\partial x\partial y}\right)-\rho_0\left(\frac{\partial^2 v_1}{\partial t\partial x}+U\frac{\partial^2 v_1}{\partial x^2}\right)$$
$$=-\cancel{\frac{\partial^2 p_1}{\partial x\partial y}}+\cancel{\frac{\partial^2 p_1}{\partial x\partial y}}+\frac{\partial \rho_1}{\partial x}g$$

$$u_1=\hat{u}(y)e^{i(kx-\omega t)}$$
$$v_1=\hat{v}(y)e^{i(kx-\omega t)}$$
$$\rho_1=\hat{\rho}(y)e^{i(kx-\omega t)}$$

Eq-n for density:

$$-i\omega\hat{\rho}+\hat{v}\rho_0'+Uik\hat{\rho}=0$$

$-ik(U-c)\hat{\rho}=\rho_0'\hat{v}$, where $c=\omega/k$

Eq-n for velocity:

$$\rho_0'(-i\omega\hat{u}+\hat{v}U'+ikU\hat{u})+\rho_0(-i\omega\hat{u}'+\hat{v}U''+ikU\hat{u}')$$
$$-\rho_0(\omega k\hat{v}-k^2U\hat{v})=ik\hat{\rho}g$$

Given that $\omega=ck$,

$$\rho_0' ik(U-c)\hat{u} + \rho_0'\hat{v}U' + \rho_0 ik(U-c)\hat{u}' + \rho_0 U''\hat{v}$$
$$+ \rho_0 k^2 (U-c)\hat{v} = ik\hat{\rho}g$$

Eliminate $\hat{u}$ from $ik\hat{u} + \hat{v}' = 0$ and divide by $-\rho_0(U-c)$:

$$\hat{v}'' + \frac{\rho_0'}{\rho_0}\hat{v}' - \left[\frac{U''}{U-c} + \frac{\rho_0'}{\rho_0}\frac{U'}{U-c} + k^2\right]\hat{v} = -ik\frac{g}{\rho_0(U-c)}\hat{\rho}$$

Eliminate $\hat{\rho}$:

$$\hat{v}'' + \frac{\rho_0'}{\rho_0}\hat{v}' - \left[-\frac{N^2}{(U-c)^2} + \frac{U''}{U-c} + \frac{\rho_0'}{\rho_0}\frac{U'}{U-c} + k^2\right]\hat{v} = 0.$$

where $N^2 = -g\frac{\rho_0'}{\rho_0}$ — buoyancy frequency

Part II. Derive the perturbation equation in the case when the density of the base flow varies with height much slower then *U(y)* and $\hat{v}(y)$.

1) $|\hat{v}''| \sim \frac{|\hat{v}|}{L^2} \rightarrow$ lengthscale of change in $\hat{v}$

2) $\left|\frac{\rho_0'}{\rho_0}\hat{v}'\right| \sim \frac{|\hat{v}|}{L} \cdot \frac{1}{L_0}$

$$L_0 >> L \implies |\hat{v}''| >> \left|\frac{\rho_0'}{\rho_0}\hat{v}'\right|$$